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PARTITIONS INTO PRIMES

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ABSTRACT. We investigate the asymptotic behavior of the partition function $p_\Lambda(n)$ defined by $\sum_{n=0}^\infty p_\Lambda(n)x^n = \prod_{m=1}^\infty (1-x^m)^{-\Lambda(m)}$, where $\Lambda(n)$ denotes the von Mangoldt function. Improving a result of Richmond, we show that $\log p_\Lambda(n) = 2\sqrt{\zeta(2)n} + O(\sqrt{n}\exp\{-c(\log n)(\log_2 n)^{-2/3}(\log_3 n)^{-1/3}\})$, where c is a positive constant and $\log_k$ denotes the k times iterated logarithm. We also show that the error term can be improved to $O(n^{1/4})$ if and only if the Riemann Hypothesis holds.

1. Introduction

The asymptotic behavior of partition functions has been extensively studied in the literature. The most famous result is the asymptotic formula

$$p(n) \sim \frac{1}{(4\sqrt{3})n} e^{\pi\sqrt{2n/3}} \qquad (n \to \infty)$$

for the ordinary partition function $p(n)$, proved in 1918 by Hardy and Ramanujan [3]. The asymptotic behavior of more general partition functions has been studied by many authors, including Ingham [4], Kohlbecker [6], Meinardus [7], [8], Roth and Szekeres [12], Schwarz [13], [14], and Richmond [10], [11].

Of particular interest are functions related to partitions into primes. As an application of an asymptotic formula for general partition functions, Roth and Szekeres [12] showed that the number $q_{\mathcal{P}}(n)$ of partitions of n into distinct primes satisfies

$$\log q_{\mathcal{P}}(n) = \pi\sqrt{\frac{2}{3}}\left(\frac{n}{\log n}\right)^{1/2}\left\{1 + O\left(\frac{\log\log n}{\log n}\right)\right\}.$$

A similar, but in some ways more natural, partition function is the function $p_\Lambda(n)$ defined by

$$\sum_{n=0}^\infty p_\Lambda(n)x^n = \prod_{m=1}^\infty (1-x^m)^{-\Lambda(m)},$$

where

$$\Lambda(m) = \begin{cases} \log p, & m = p^r, \\ 0, & \text{else}, \end{cases}$$

is the von Mangoldt function. The function $p_\Lambda(n)$ represents a weighted count of the number of partitions of n into prime powers. This weighted partition function was first introduced and studied in 1950 by Brigham [1], who proved a conditional

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result under the assumption of the Riemann Hypothesis. More recently, Richmond [11], using an asymptotic formula for general partition functions (see [10]) and Vinogradov's zero-free region for the Riemann zeta function, obtained the following unconditional result.

Theorem A (Richmond). *There exists a positive constant c such that for all sufficiently large n*

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2}\left(1 + O\left\{\exp(-c(\log n)^{4/7}(\log\log n)^{-3/7})\right\}\right). \tag{1.1}$$

Richmond also proved a conditional result.

Theorem B (Richmond). *Let θ be the least upper bound for the real parts of the non-trivial zeros of the Riemann zeta function. Then*

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2} + O(n^{\theta/2}). \tag{1.2}$$

In particular, if the Riemann Hypothesis is true, then

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2} + O(n^{1/4}). \tag{1.3}$$

In our first result we show that the error term in Richmond's unconditional result (1.1) can be substantially improved.

Theorem 1. *There exists a positive constant c such that for all sufficiently large n*

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2}\left\{1 + O\left(\exp\left\{-\frac{c\log n}{(\log_2 n)^{2/3}(\log_3 n)^{1/3}}\right\}\right)\right\}, \tag{1.4}$$

where $\log_k$ denotes the k times iterated logarithm.

Note that the exponential factor in (1.4) is much smaller than the corresponding factor in the error term of the prime number theorem obtained by using the Vinogradov zero-free region, namely $\exp(-(\log n)^{3/5-\epsilon})$. The reason for this unexpectedly small error term lies in the fact that $\log p_\Lambda(n)$ behaves in many respects more like the power series $f(x) = \sum_{n=1}^{\infty}\Lambda(n)x^n$ than the partial sum $\Psi(u) = \sum_{n\le u}\Lambda(n)$; indeed, it would not be hard to show that, as $x \to 1-$, $f(x)$ differs from its approximation $1/(1-x)$ by a similarly small error term.

While the estimate of Theorem A can be substantially improved, our next result shows that Theorem B is best possible.

Theorem 2. *We have*

$$\log p_\Lambda(n) - 2\sqrt{\zeta(2)}n^{1/2} = \Omega_\pm(n^{1/4}).$$

Our final result gives a converse to Theorem B.

Theorem 3. *Let θ be the least upper bound for the real parts of the zeros of the Riemann zeta function, and let θ' be the greatest lower bound for all α for which*

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2} + O(n^{\alpha/2})$$

as $n \to \infty$. Then $\theta \le \theta'$.

Combining this result with Theorem B yields the following corollary.

Corollary. *Let θ and θ' be defined as in Theorem 3. Then $\theta = \theta'$. In particular, the Riemann Hypothesis is true if and only if*

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2} + O_\epsilon(n^{1/4+\epsilon})$$

for all $\epsilon > 0$.

Let

$$P_\Lambda(u) = \sum_{n\le u} p_\Lambda(n)$$

be the summatory function of $p_\Lambda(n)$. In Section 4 we show that Theorems 1-3 are true if and only if the corresponding statements with n replaced by u and $p_\Lambda(n)$ replaced by $P_\Lambda(u)$ are true. Hence it suffices to prove Theorems 1-3 with $P_\Lambda(u)$ instead of $p_\Lambda(n)$.

Our main tools for proving these results are Abelian and Tauberian theorems that relate estimates for $P_\Lambda(u)$ to estimates for the Laplace-Stieltjes transform of $P_\Lambda(u)$, defined by

$$F_\Lambda(x) = \int_0^\infty e^{-xu}\, dP_\Lambda(u).$$

We will state and prove these results in Section 2. In Section 3 we establish some lemmas relating the behavior of general weighted partition functions $p_w(n)$ defined by

$$\sum_{n=0}^\infty p_w(n)x^n = \prod_{m=1}^\infty (1-x^m)^{-w(m)}$$

to analytic properties of the Dirichlet series

$$f_w(s) = \sum_{n=1}^\infty \frac{w(n)}{n^s},$$

where $w(m)$ is a non-negative function defined on the set of positive integers. Our methods here are, to some extent, similar to those in Meinardus [7]. However, Meinardus made much stronger assumptions on the analytic properties of $f_w(s)$. These assumptions are not satisfied in the case $w(n) = \Lambda(n)$, so that Meinardus' results are not applicable to $p_\Lambda(n)$. In Section 4 we complete the proof of Theorems 1-3. In Section 5 we use one of our Tauberian results to give a new proof of Theorem B that is simpler and more elementary than the original proof of Richmond.

2. Some Abelian and Tauberian Results

Our first result is an elementary Abelian result which generalizes and extends a result of Freiman [2, p. 276].

Proposition 1. *Suppose that $P(u)$ is a non-negative and non-decreasing function satisfying $P(t) = O_\epsilon(e^{\epsilon t})$ for every $\epsilon > 0$. For $x > 0$ let*

$$F(x) = \int_0^\infty e^{-xu}\, dP(u)$$

be the Laplace-Stieltjes transform of $P(u)$. Suppose that for some constants $A > 0$ and $0 < a < 1$ the inequality

$$\log P(u) \ge Au^a - r(u) \tag{2.1a}$$

holds for all sufficiently large u, where $r(u)$ is a positive differentiable function satisfying

$$r(u)u^{-a} \to 0 \text{ monotonically as } u \to \infty, \tag{R1}$$

and

$$r(u)/(\log u) \to \infty \text{ monotonically as } u \to \infty. \tag{R2}$$

Then

$$\log F(x) \geq Bx^{-b} - r(B'x^{-b-1}) \tag{2.2a}$$

for all sufficiently small x, where

$$B = (Aa)^{1/(1-a)}(1-a)/a, \qquad b = a/(1-a), \qquad B' = Bb. \tag{2.3}$$

Similarly, if

$$\log P(u) \leq Au^a + r(u) \tag{2.1b}$$

for all sufficiently large u, then with the same notation

$$\log F(x) \leq Bx^{-b} + 2r(B'x^{-b-1}) \tag{2.2b}$$

for all sufficiently small x.

As a simple consequence of Proposition 1, we have the following corollary.

Corollary. *Let $P(u)$, $F(x)$, A, a, B, b and B' be given as in Proposition 1. Suppose that $r(u)$ satisfies conditions* (R1) *and* (R2). *Then*

$$\log F(x) - Bx^{-b} = \Omega_+(r(B'x^{-b-1}))$$

as $x \to 0$ implies that

$$\log P(u) - Au^a = \Omega_+(r(u))$$

as $u \to \infty$. The same statement holds if Ω_+ is replaced by Ω_-.

Proof of Proposition 1. Suppose first that (2.1a) holds for sufficiently large u. By the monotonicity of $P(t)$ and the assumption that $P(t)$ is non-negative, we have for all $x > 0$ and all $u > 0$,

$$\begin{aligned} F(x) &= \int_0^\infty e^{-xt}\,dP(t) = x\int_0^\infty e^{-xt}P(t)\,dt \\ &\geq x\int_u^\infty e^{-xt}P(t)\,dt \geq x\int_u^\infty e^{-xt}P(u)\,dt = e^{-xu}P(u). \end{aligned} \tag{2.4}$$

(Note that the convergence of the integral defining $F(x)$ is ensured by the assumption that $P(t) = O_\epsilon(e^{\epsilon t})$ for every $\epsilon > 0$.)

By (2.1a) it follows that

$$F(x) \geq \exp\{-xu + Au^a - r(u)\} \tag{2.5}$$

for all sufficiently large u and all $x > 0$. We choose $u = u_x$ to maximize $-xu + Au^a$, i.e., we let u_x be defined by

$$x = Aau_x^{-(1-a)}. \tag{2.6}$$

If x is sufficiently small, then u_x will be large enough for (2.5) to hold. We note that

$$xu_x = x(Aax^{-1})^{1/(1-a)} = B'x^{-a/(1-a)} = B'x^{-b} \tag{2.7}$$

and

$$Au_x^a = A(Aax^{-1})^{a/(1-a)} = (Aa)^{1/(1-a)}a^{-1}x^{-b} = B'a^{-1}x^{-b}. \tag{2.8}$$

Substituting these expressions into the right-hand side of (2.5), we obtain

$$\log F(x) \geq B'(-1+a^{-1})x^{-b} - r(B'x^{-b-1}) = Bx^{-b} - r(B'x^{-b-1}),$$

which proves (2.2a). It remains to prove that (2.1b) implies (2.2b).

Let u_0 be a positive constant such that (2.1b) and conditions (R1)-(R2) are satisfied for $u \geq u_0$. Given a positive number x, we let u_x be defined by (2.6) and assume that x is small enough so that $u_x \geq 2u_0$. We write

$$\begin{aligned} F(x) &= x\left\{\int_0^{u_0} + \int_{u_0}^{(1-\mu)u_x} + \int_{(1-\mu)u_x}^{(1+\mu)u_x} + \int_{(1+\mu)u_x}^{\infty}\right\} e^{-xt}P(t)\,dt \\ &= x\{I_1 + I_2 + I_3 + I_4\}, \end{aligned} \tag{2.9}$$

where μ is a parameter to be chosen so that xI_3 has the same order of magnitude as $F(x)$. It turns out that the optimal choice for μ is

$$\mu = \mu(u_x) = K\sqrt{u_x^{-a}r(u_x)}, \tag{2.10}$$

where K is a positive constant to be chosen later in terms of A and a. We note that condition (R1) implies that $\mu(u_x) \to 0$ as $u_x \to \infty$. In particular, we have $0 < \mu < 1/2$ if x is sufficiently small in terms of K, which we will henceforth assume.

The integral I_1 is bounded by $\int_0^{u_0} P(t)\,dt$ and thus of order $O(1)$. To estimate I_4, we define $\phi(t)$ for $t \geq u_0$ by

$$\phi(t) = -xt + At^a + r(t).$$

Since, by condition (R1), $r(t)t^{-a}$ is monotonically decreasing to 0, we have $r'(t)t^{-a} - ar(t)t^{-a-1} \leq 0$ and $r(t)t^{-1} \leq r(u_x)u_x^{-a}u_x^{-(1-a)} = K^{-2}\mu^2u_x^{-(1-a)}$ for all $t \geq u_x$. Thus, for all $t \geq (1+\mu)u_x$ we obtain

$$\begin{aligned} \phi'(t) &= -x + Aat^{-(1-a)} + r'(t) \leq -x + Aat^{-(1-a)} + ar(t)t^{-1} \\ &\leq -x + Aa(1+\mu)^{-(1-a)}u_x^{-(1-a)} + K^{-2}a\mu^2u_x^{-(1-a)}. \end{aligned}$$

By (2.6), the last expression is equal to $(-1 + (1+\mu)^{-(1-a)} + O(\mu^2))x$. Since $(1+\mu)^{-(1-a)} = 1-(1-a)\mu+O(\mu^2)$, we have $-1+(1+\mu)^{-(1-a)}+O(\mu^2) \leq -(1-a)\mu/2$ for all sufficiently small x. Hence for all sufficiently small x and $t \geq (1+\mu)u_x$ we obtain

$$\phi'(t) \leq -\frac{1-a}{2}x\mu.$$

It follows that

$$\begin{aligned} xI_4 &\leq x\int_{(1+\mu)u_x}^{\infty} e^{\phi(t)}\,dt \leq -\frac{2}{(1-a)\mu}\int_{(1+\mu)u_x}^{\infty} e^{\phi(t)}\phi'(t)\,dt = \frac{2}{(1-a)\mu}e^{\phi((1+\mu)u_x)} \\ &= \frac{2}{(1-a)\mu}\exp\left\{-x(1+\mu)u_x + A(1+\mu)^a u_x^a + r((1+\mu)u_x)\right\}. \end{aligned}$$

We substitute (2.7) and (2.8) into the last expression and note that condition (R1) implies that $r((1+\mu)u_x)((1+\mu)u_x)^{-a} \leq r(u_x)u_x^{-a}$, i.e.,

$$r((1+\mu)u_x) \leq (1+\mu)^a r(u_x) \tag{2.11}$$

for all sufficiently small x. This yields

$$\begin{aligned}
xI_4 &\le \frac{2}{(1-a)\mu}\exp\left\{B'x^{-b}\left(-(1+\mu)+(1+\mu)^a a^{-1}+(1+\mu)^a(Aa)^{-1}r(u_x)u_x^{-a}\right)\right\}\\
&\le \exp\big\{B'x^{-b}\big((a^{-1}-1)+\frac{1}{2}(a-1)\mu^2+O(\mu^3)+(1+\mu)^a(Aa)^{-1}r(u_x)u_x^{-a}\big)\\
&\qquad\quad +O(\log\mu^{-1})\big\}\\
&= \exp\big\{Bx^{-b}\big(1-\frac{1}{2}a\mu^2+O(\mu^3)+(1+O(\mu))A^{-1}(1-a)^{-1}r(u_x)u_x^{-a}\big)\\
&\qquad\quad +O(\log\mu^{-1})\big\},
\end{aligned}$$

where we have used the identity $B'(a^{-1}-1)=B$ which follows from (2.3). Expressing $r(u_x)u_x^{-a}$ in terms of μ using (2.10), the last expression becomes

$$\exp\left\{Bx^{-b}\left(1-\mu^2\left(\frac{a}{2}-K^{-2}A^{-1}(1-a)^{-1}\right)+O(\mu^3)+O(x^b\log\mu^{-1})\right)\right\}.$$

By (2.8), (2.10) and condition (R2) we have

$$x^b\log\mu^{-1}\ll u_x^{-a}\log u_x=o(u_x^{-a}r(u_x))=o(\mu^2).$$

Choosing now K sufficiently large in terms of A and a yields

$$xI_4\le\exp\left\{Bx^{-b}(1-a\mu^2/4)\right\} \tag{2.12}$$

for all sufficiently small x. By a similar argument we have

$$xI_2\le\exp\left\{Bx^{-b}(1-a\mu^2/4)\right\} \tag{2.13}$$

for all sufficiently small x.

We now estimate I_3. By (2.1b) we have

$$xI_3=x\int_{(1-\mu)u_x}^{(1+\mu)u_x}e^{-xt}P(t)\,dt\le x\int_{(1-\mu)u_x}^{(1+\mu)u_x}e^{-xt+At^a+r(t)}\,dt$$

We observe that $t=u_x$ maximizes $-xt+At^a$ and, by condition (R2), $r(t)\le r((1+\mu)u_x)$ for all $t\le(1+\mu)u_x$. Hence, by (2.7), (2.8) and (2.11),

$$\begin{aligned}
xI_3 &\le 2x\mu u_x\exp\{-xu_x+Au_x^a+r((1+\mu)u_x)\}\\
&\le 2x\mu u_x\exp\left\{Bx^{-b}+(1+\mu)^a r(B'x^{-b-1})\right\}
\end{aligned} \tag{2.14}$$

for all sufficiently small x. Combining (2.9), (2.12), (2.13) and (2.14), and noting that, by (2.7) and (2.10),

$$\log(x\mu u_x)=\log\left(B'x^{-b}K\sqrt{u_x^{-a}r(u_x)}\right)=O\big(\log(u_x^a r(u_x))\big)=O(\log x^{-1}),$$

we finally obtain

$$\begin{aligned}
F(x)&\le O(x)+O(\exp\left\{Bx^{-b}-a\mu^2/4\right\})\\
&\qquad +\exp\left\{Bx^{-b}+(1+\mu)^a r(B'x^{-b-1})+O(\log x^{-1})\right\}\\
&=\exp\left\{Bx^{-b}+(1+\mu)^a r(B'x^{-b-1})+O(\log x^{-1})\right\}\\
&\le\exp\left\{Bx^{-b}+2r(B'x^{-b-1})\right\}
\end{aligned}$$

for all sufficiently small x, where in the last step we have used the assumption (R2). This yields the desired estimate (2.2b). □

Our next result is a Tauberian counterpart to Proposition 1.

Proposition 2. *Let $P(u)$ and $F(x)$ be defined as in Proposition 1. Suppose that for some constants $B, b > 0$ and $B' = Bb$ the inequality*

$$\left|\log F(x) - Bx^{-b}\right| \le r(B'x^{-b-1}) \tag{2.15}$$

holds for all sufficiently small x, where $r(u)$ is a positive differentiable function satisfying conditions (R1) *and* (R2) *in Proposition 1 with $a = b/(1+b)$. Then*

$$-C\sqrt{u^a r(u)} \le \log P(u) - Au^a \le r(u) \tag{2.16}$$

for all sufficiently large u, where A and a are determined by (2.3) *and C is a constant depending on B and b.*

Proof. Suppose that (2.15) holds for all sufficiently small x. We first prove the upper bound for $\log P(u)$.

Given a positive number u, by (2.4) and assumption (2.15) we have

$$P(u) \le e^{xu}F(x) \le \exp\left\{xu + Bx^{-b} + r(B'x^{-b-1})\right\} \tag{2.17}$$

for all sufficiently small x. We choose

$$x = x_u = Aau^{-(1-a)}, \tag{2.18}$$

where A and a are determined by (2.3), and note that with this choice of x we have $u = u_x$, where u_x is given by (2.6). We assume that u is large enough so that (2.17) holds for $x = x_u$. Using (2.7) and (2.8) with x_u and u in place of x and u_x, we see that

$$x_u u + Bx_u^{-b} = B'x_u^{-b}(1 + b^{-1}) = Aau_x^a\left(1 + \frac{1-a}{a}\right) = Au_x^a \tag{2.19}$$

and

$$B'x_u^{-b-1} = u. \tag{2.20}$$

Thus, we obtain the upper bound

$$P(u) \le \exp\{x_u u + Bx_u^{-b} + r(B'x_u^{-b-1})\} = \exp\{Au^a + r(u)\} \tag{2.21}$$

for all sufficiently large u. This implies the upper bound in (2.16). It remains to prove the lower bound.

Let u_0 be a large positive constant such that (2.21) holds for $u \ge u_0$. Assuming that $u \ge 2u_0$, we split the integral defining $F(x_u)$ into four parts as before. Let I_1, I_2, I_3 and I_4 be defined by (2.9) with $x = x_u$ and u_x replaced by u. We let $\mu = \mu(u) = K\sqrt{u^{-a}r(u)}$ be defined as in (2.10), where K is a large positive constant to be chosen later. Using the upper bound (2.21) and arguing exactly as in the proof of Proposition 1, we see that the upper bounds (2.12) and (2.13) for I_4 and I_2 remain valid for sufficiently large u with x_u and u in place of x and u_x. We note that, by (2.3), (2.8) and the definition of μ,

$$\frac{1}{4}Ba\mu^2 x_u^{-b} = \frac{1}{4}BaK^2 r(u)u^{-a}x_u^{-b} = \frac{1}{4}BaK^2 r(u)\frac{Aa}{B'} = \frac{1}{4}Aa(1-a)K^2 r(B'x_u^{-b-1}).$$

We now choose the constant K large enough so that $Aa(1-a)K^2/4 \ge 2$. Then by (2.12) and assumption (2.15), we have

$$\begin{aligned} x_u I_4 &\le \exp\left\{Bx_u^{-b} - \frac{1}{4}Ba\mu^2 x_u^{-b}\right\} \le \exp\left\{Bx_u^{-b} - 2r(B'x_u^{-b-1})\right\} \\ &\le F(x_u)\exp\left\{-r(B'x_u^{-b-1})\right\} = o(F(x_u)). \end{aligned}$$

Similarly, we see that $x_uI_2 = o(F(x_u))$. Hence $x_uI_3 \geq F(x_u)(1 + o(1))$. On the other hand, bounding the integral I_3 trivially, we obtain

$$x_uI_3 \leq 2\mu u x_u e^{-x_u(1-\mu)u} P((1+\mu)u).$$

It follows that

$$P((1+\mu)u) \geq F(x_u) \exp\{x_u(1-\mu)u - \log(2\mu u x_u) + o(1)\}.$$

By (2.10), (2.18) and conditions (R1) and (R2) we have

$$\log(2\mu u x_u) \ll \log u \ll r(u).$$

Thus, using the bound (2.15) for $F(x_u)$, we obtain

$$P((1+\mu)u) \geq \exp\left\{Bx_u^{-b} - r(B'x_u^{-b-1}) + x_u(1-\mu)u + O(r(u))\right\}.$$

Substituting (2.18), (2.19) and (2.20) into the last expression, we see that

$$\log P((1+\mu)u) \geq Au^a - Aa\mu u^a + O(r(u)) = A(1+\mu)^a u^a + O(\mu u^a) + O(r(u)).$$

The last error term can be omitted since $r(u) = o(\sqrt{r(u)u^a}) = o(\mu u^a)$ by condition (R1) and (2.10). Moreover, by condition (R2), we have $\mu u^a = K\sqrt{r(u)u^a} \leq K\sqrt{r((1+\mu)u)(1+\mu)^a u^a}$. It follows that

$$\log P((1+\mu)u) \geq A(1+\mu)^a u^a - C\sqrt{r((1+\mu)u)(1+\mu)^a u^a}$$

for all sufficiently large u. Since $(1+\mu(u))u$ is a continuous function of u and tends to infinity when $u \to \infty$, for all sufficiently large v, there exists a u such that $v = (1+\mu(u))u$. Hence we have

$$\log P(v) \geq Av^a - C\sqrt{v^a r(v)}$$

for all sufficiently large v. This completes the proof of the proposition. □

Proposition 2 is not a complete converse to Proposition 1, as the lower bound in (2.16) is weaker than the lower bound for $\log P(u)$ in Proposition 1. Our next result gives, under a stronger hypothesis, a complete converse.

Proposition 3. *Let $P(u)$ and $F(x)$ be defined as in Proposition 1. Suppose that for some constants B, $b > 0$ and $B' = Bb$ the inequality*

$$\left|\log F(x) - Bx^{-b}\right| \leq r(B'x^{-b-1}) \tag{2.15}$$

holds for all sufficiently small x, where $r(u)$ is a positive differentiable function satisfying conditions (R1) *and* (R2) *in Proposition 1 with $a = b/(1+b)$. Suppose that, in addition, $r(u)$ satisfies*

$$r(u) \gg u^{a/2} \tag{R3}$$

as $u \to \infty$. Suppose further that the function $G(x) = \log F(x)$ satisfies

$$G'(x) = -B'x^{-b-1} + O\left(x^{-1}r(B'x^{-b-1})\right) \tag{2.22}$$

and

$$G''(x) \gg x^{-b-2} \tag{2.23}$$

as $x \to 0$. Then

$$-Cr(u) \leq \log P(u) - Au^a \leq r(u) \tag{2.24}$$

for all sufficiently large u, where A and a are determined by (2.3)*, and C is a positive constant depending on B, b and the constants implicit in* (2.22)*,* (2.23) *and condition* (R3)*.*

Proof. The upper bound in (2.24) follows from Proposition 2. Therefore it remains to prove the lower bound in (2.24). To this end we use a method of Odlyzko [9].

Given a large number u, we let $0 < \mu_1 < \mu_2 < 1$ be positive numbers to be chosen later as functions of u. Set $u_0 = u$, $u_1 = (1-\mu_1)u$, $u_2 = (1-\mu_2)u$, and let x_0, x_1 and x_2 be defined by (2.18) with $u = u_0$, u_1 and u_2 respectively. Thus, we have $x_1 = (1-\mu_1)^{-(1-a)}x_0$ and $x_2 = (1-\mu_2)^{-(1-a)}x_0$. We consider the function

$$\begin{aligned} h(t) = \exp\{x_1u_1 - x_1t\} &- \exp\{x_0u_0 + x_1u_1 - x_1u_0 - x_0t\} \\ &- \exp\{x_2u_2 + x_1u_1 - x_1u_2 - x_2t\}. \end{aligned}$$

Since $x_0 < x_1 < x_2$, we have for $t \geq u_0$

$$x_1u_1 - x_1t \leq x_1u_1 - x_1t + (x_1-x_0)(t-u_0) = x_0u_0 + x_1u_1 - x_1u_0 - x_0t,$$

and for $t \leq u_2$

$$x_1u_1 - x_1t \leq x_1u_1 - x_1t + (x_2-x_1)(u_2-t) = x_2u_2 + x_1u_1 - x_1u_2 - x_2t.$$

Thus $h(t) \leq 0$ for $t \leq u_2$ and $t \geq u_0$. Now we let

$$\begin{aligned} H &= \int_0^\infty h(t)\,dP(t) \\ &= e^{x_1u_1}F(x_1) - e^{x_0u_0+x_1u_1-x_1u_0}F(x_0) - e^{x_2u_2+x_1u_1-x_1u_2}F(x_2). \end{aligned}$$

Then we see that

$$\begin{aligned} H &\leq (P(u_0) - P(u_2)) \times \max_{u_2\leq t\leq u_0} h(t) \\ &\leq P(u_0) \times \max_{u_2\leq t\leq u_0} e^{x_1u_1-x_1t} = P(u_0)e^{x_1(u_1-u_2)}. \end{aligned}$$

Hence, we have

$$\begin{aligned} P(u_0) \geq \big\{&e^{x_1u_1}F(x_1) - e^{x_0u_0+x_1u_1-x_1u_0}F(x_0) \\ &- e^{x_2u_2+x_1u_1-x_1u_2}F(x_2)\big\}e^{-x_1(u_1-u_2)}. \end{aligned} \tag{2.25}$$

We now show that, with a suitable choice of μ_1 and μ_2, $e^{x_0u_0+x_1u_1-x_1u_0}F(x_0) + e^{x_2u_2+x_1u_1-x_1u_2}F(x_2) \leq e^{x_1u_1}F(x_1)/2$, and thus the last expression has the same order of magnitude as $F(x_1)e^{x_1u_2}$.

By assumptions (2.22) and (2.23), there are positive constants C_1 and C_2 such that

$$-B'x^{-b-1} - C_1x^{-1}r(B'x^{-b-1}) \leq G'(x) \leq -B'x^{-b-1} + C_1x^{-1}r(B'x^{-b-1})$$

and

$$G''(x) \geq C_2x^{-b-2}$$

when x is sufficiently small. Thus, Taylor's formula yields that

$$\begin{aligned} G(x_1) - G(x_0) &\geq G'(x_0)(x_1-x_0) + \frac{1}{2}\min_{x_0\leq\xi\leq x_1} G''(\xi)(x_1-x_0)^2 \\ &\geq (x_1-x_0)(-B'x_0^{-b-1} - C_1x_0^{-1}r(B'x_0^{-b-1})) \\ &\quad + \frac{1}{2}C_2x_1^{-b-2}(x_1-x_0)^2. \end{aligned} \tag{2.26}$$

We now choose μ_1 to be of the form $\mu_1 = Kx_0^b r(B'x_0^{-b-1})$, where K is a large constant independent of u_0. We note that, by (2.8) and (2.20), μ_1 is also equal to $KB'(Aa)^{-1}u_0^{-a}r(u_0)$. Thus, by condition (R3), we have

$$\mu_1 \geq KC_3 u_0^{-a/2} \tag{2.27}$$

for all sufficiently large u_0, where C_3 is a positive constant depending on the constant implicit in condition (R3). Moreover, by condition (R1), this choice of μ_1 satisfies $\mu_1 \to 0$ as $u_0 \to \infty$. It follows that when u_0 is sufficiently large,

$$x_1 - x_0 = x_0((1-\mu_1)^{-(1-a)} - 1) \begin{cases} \leq 2(1-a)\mu_1 x_0, \\ \geq (1-a)\mu_1 x_0. \end{cases}$$

Hence, recalling that, by (2.20), $B'x_0^{-b-1} = u_0$, we have from (2.26)

$$\begin{aligned} G(x_1) - G(x_0) \geq& -(x_1 - x_0)u_0 - 2C_1(1-a)\mu_1 r(u_0) \\ &\qquad + \frac{1}{2}C_2 x_0^{-b-2}(1-\mu_1)^{(1-a)(b+2)}(1-a)^2\mu_1^2 x_0^2 \\ \geq& -(x_1 - x_0)u_0 - 2C_1(1-a)\mu_1 r(u_0) + \frac{1}{4}C_2(1-a)^2\mu_1^2 x_0^{-b} \end{aligned}$$

for all sufficiently large u_0. We now choose K sufficiently large in terms of B, b, C_1 and C_2 so that the second term on the right-hand side of the last expression is less than one half of the last term uniformly for all sufficiently large u_0, and so that the last term is at least $2\log 4$ (which is possible by (2.8) and (2.27)), i.e., so that

$$2C_1(1-a)\mu_1 r(B'x_0^{-b-1}) \leq \frac{1}{8}C_2(1-a)^2\mu_1^2 x_0^{-b}$$

and

$$\frac{1}{8}C_2(1-a)^2\mu_1^2 x_0^{-b} \geq \log 4$$

for all sufficiently large u_0. Then it follows that

$$\begin{aligned} G(x_1) - G(x_0) &\geq -(x_1 - x_0)u_0 - 2C_1(1-a)\mu_1 r(B'x_0^{-b-1}) + \frac{1}{4}C_2(1-a)^2\mu_1^2 x_0^{-b} \\ &\geq -(x_1 - x_0)u_0 + \frac{1}{8}C_2(1-a)^2\mu_1^2 x_0^{-b} \\ &\geq -(x_1 - x_0)u_0 + \log 4. \end{aligned}$$

Hence

$$e^{x_1 u_1}F(x_1) \geq 4e^{x_0 u_0 + x_1 u_1 - x_1 u_0}F(x_0) \tag{2.28}$$

for all sufficiently large u_0. Similarly, if we choose $\mu_2 = 2\mu_1$, then

$$e^{x_1 u_1}F(x_1) \geq 4e^{x_2 u_2 + x_1 u_1 - x_1 u_2}F(x_2) \tag{2.29}$$

for all sufficiently large u_0. Therefore, by (2.25), (2.28) and (2.29), we have for all sufficiently large u_0

$$P(u_0) \geq \frac{1}{2}e^{x_1 u_1}F(x_1)e^{-(u_1 - u_2)x_1} = \frac{1}{2}F(x_1)e^{x_1 u_2} = \exp\left\{G(x_1) + x_1 u_2 + O(1)\right\}.$$

On the other hand, by (2.15) and the definition of x_1 and u_2, we have

$$\begin{aligned}G(x_1) + x_1 u_2 &\ge B x_1^{-b} - r(B' x_1^{-b-1}) + x_1 u_2 \\ &= B(1-\mu_1)^{(1-a)b} x_0^{-b} - r(B' x_1^{-b-1}) + (1-\mu_1)^{-(1-a)} x_0 (1-2\mu_1) u_0 \\ &= B x_0^{-b} + x_0 u_0 + O(\mu_1 x_0^{-b}) - r(B' x_1^{-b-1}) \\ &= B x_0^{-b} + x_0 u_0 + O(r(B' x_0^{-b-1})) - r(B' x_1^{-b-1})\end{aligned}$$

for all sufficiently large u_0. By condition (R2), we have $r(B' x_1^{-b-1}) \le r(B' x_0^{-b-1})$. It follows that, by (2.19) and (2.20),

$$\log P(u_0) \ge G(x_1) + x_1 u_2 + O(1) \ge B x_0^{-b} + x_0 u_0 - Cr(B' x_0^{-b-1}) = A u_0^a - Cr(u_0)$$

for all sufficiently large u_0, where C is a constant depending on C_1, C_2, B and b. This completes the proof of the proposition. □

3. Some Lemmas on Dirichlet Series and Mellin Transforms

Let $w(n)$ be a non-negative function defined on the set of positive integers. Let

$$f_w(s) = \sum_{n=1}^{\infty} \frac{w(n)}{n^s} \tag{3.1}$$

be the Dirichlet series generated by $w(n)$, and suppose that $f_w(s)$ has finite abscissa of absolute convergence σ_a. We define $F_w(x)$ for $x > 0$ by

$$F_w(x) = \prod_{n=1}^{\infty} (1 - e^{-nx})^{-w(n)}.$$

We note that in the case $w(n) = \Lambda(n)$, we have $f_w(s) = -\zeta'(s)/\zeta(s)$, and $F_w(x)$ is the generating function for $p_\Lambda(n)$.

In this section we will prove some lemmas relating analytic properties of $f_w(s)$ to the behavior of $F_w(x)$.

Lemma 1. *For any $\kappa > \max(0, \sigma_a)$ we have*

$$\log F_w(x) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \Gamma(s)\zeta(1+s) f_w(s) x^{-s}\, ds. \tag{3.2}$$

Furthermore, for any $T > 1$ we have

$$\begin{aligned}\log F_w(x) = &\frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \Gamma(s)\zeta(1+s) f_w(s) x^{-s}\, ds \\ &+ O\left(\zeta(1+\kappa) f_w(\kappa) x^{-\kappa} T^{\kappa+1/2} e^{-\pi T/2}\right),\end{aligned} \tag{3.3}$$

where the O-constant is absolute.

Proof. The proof of (3.2) can be found in Meinardus [7, p. 390].

To prove (3.3), we observe that, by Stirling's formula,

$$\begin{aligned}\frac{1}{2\pi i} &\left\{ \int_{\kappa+iT}^{\kappa+i\infty} + \int_{\kappa-i\infty}^{\kappa-iT} \right\} \Gamma(s)\zeta(1+s) f_w(s) x^{-s}\, ds \\ &\ll \zeta(1+\kappa) f_w(\kappa) x^{-\kappa} \int_T^{\infty} t^{\kappa+1/2} e^{-\pi t/2}\, dt \\ &\ll \zeta(1+\kappa) f_w(\kappa) x^{-\kappa} T^{\kappa+1/2} e^{-\pi T/2}.\end{aligned}$$

Thus (3.3) follows. □

Lemma 2. *For $0 < x < 1$ and $\operatorname{Re} s > \max(0, \sigma_a)$ we have*

$$\int_0^1 x^{s-1} \log F_w(x)\, dx = \Gamma(s)\zeta(1+s) f_w(s) - h_w(s),$$

where $h_w(s)$ is an entire function.

Proof. Suppose that s satisfies $\operatorname{Re} s > \max(0, \sigma_a)$. Then

$$\begin{aligned}\int_0^1 x^{s-1} \log F_w(x)\, dx &= \left\{\int_0^\infty - \int_1^\infty\right\} x^{s-1} \log F_w(x)\, dx \\ &= \left\{\int_0^\infty - \int_1^\infty\right\} x^{s-1} \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{w(n)}{k} e^{-knx}\, dx \\ &= \sum_{k=1}^\infty \sum_{n=1}^\infty \left\{\frac{\Gamma(s) w(n)}{n^s k^{s+1}} - \frac{w(n)}{k} \int_1^\infty x^{s-1} e^{-knx} dx\right\} \\ &= \Gamma(s)\zeta(1+s) f_w(s) - h_w(s),\end{aligned}$$

where interchanging the order of integration and summation is justified by the absolute convergence of the double series involved. The function $h_w(s)$ here is given by

$$h_w(s) = \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{w(n)}{k} \int_1^\infty x^{s-1} e^{-knx}\, dx = \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{w(n)}{k^{1+s} n^s} \Gamma(s, kn),$$

where $\Gamma(s,u) = \int_u^\infty x^{s-1} e^{-x}\, dx$ is the incomplete Gamma function. Using the estimate

$$|\Gamma(s,u)| \ll_{\sigma_1,\sigma_2} \int_u^\infty e^{-x/2}\, dx \ll_{\sigma_1,\sigma_2} e^{-u/2} \qquad (\sigma_1 \le \operatorname{Re} s \le \sigma_2,\ u \ge 1)$$

and the bound $w(n) \ll n^{\sigma_a+1}$ (which follows from the convergence of $f_w(s)$ at $s = \sigma_a + 1$), we see that the double series defining $h_w(s)$ converges uniformly in any strip $\sigma_1 \le \operatorname{Re} s \le \sigma_2$, and hence represents an entire function of s. $\square$

Lemma 3 (Landau). *(i) Let $g(n)$ be a function defined on the set of positive integers and of constant sign for all sufficiently large n. Suppose that the Dirichlet series*

$$\sum_{n=1}^\infty g(n) n^{-s}$$

has finite abscissa of absolute convergence σ_a. Then $s = \sigma_a$ is a singularity of the function represented by the Dirichlet series.

(ii) Let $g(x)$ be an integrable function on $[0,1]$ and of constant sign for all sufficiently small x. Suppose that the Dirichlet integral

$$\int_0^1 g(x) x^{s-1}\, dx$$

has finite abscissa of absolute convergence σ_a. Then $s = \sigma_a$ is a singularity of the function represented by the Dirichlet integral.

Proof. Part (i) is the classical version of a well-known theorem of Landau (see, e.g., Ingham [5, p. 88]). To obtain part (ii), we note that the given integral can be written as

$$\int_1^\infty h(u)u^{-s}\,du$$

with $h(u) = g(1/u)/u$ and $x = 1/u$. By the integral version of Landau's lemma (Ingham [5, p. 88]) the last integral has a singularity at $s = \sigma_a$. $\square$

Lemma 4. *Assume that for some constants $B > 0$ and $b > 0$*

$$\log F_w(x) = Bx^{-b}\{1 + o(1)\} \tag{3.4}$$

as $x \to 0+$. Then the Dirichlet series $f_w(s)$ has abscissa of convergence b and satisfies

$$f_w(s) = \frac{1}{s-b}\left\{\frac{B}{\Gamma(b)\zeta(1+b)} + o\left(\frac{|s-b|}{\sigma - b}\right)\right\} \tag{3.5}$$

as $s \to b$, while $\sigma = \operatorname{Re} s > b$.

Proof. By Lemma 2 we have

$$f_w(s) = \frac{1}{\Gamma(s)\zeta(1+s)}\left\{\int_0^1 x^{s-1}\log F_w(x)\,dx + h_w(s)\right\}, \tag{3.6}$$

where $h_w(s)$ is an entire function. On the other hand, the hypothesis (3.4) implies that

$$\begin{aligned}\int_0^1 x^{s-1}\log F_w(x)\,dx &= B\int_0^1 x^{s-b-1}\,dx + o\left(\int_0^1 x^{\sigma-b-1}\,dx\right)\\ &= \frac{B}{s-b} + o\left(\frac{1}{\sigma-b}\right)\end{aligned} \tag{3.7}$$

as $\sigma \to b+$. Combining these two estimates yields (3.5).

To show that the Dirichlet series $f_w(s)$ has abscissa of absolute convergence b, we observe that, by (3.6) and (3.7), $f_w(s)$ is analytic on the half-plane $\{s : \operatorname{Re} s > b\}$, but has a singularity at the real point $s = b$. Since $w(n)$ is non-negative, Lemma 3 implies that $f_w(s)$ has abscissa of absolute convergence b. $\square$

Lemma 5. *Suppose that*

$$g_w(s) = f_w(s) - \frac{B}{\Gamma(b)\zeta(1+b)}\frac{1}{s-b}$$

can be meromorphically continued to a half-plane $\{s : \operatorname{Re} s > \sigma_0\}$, where $\sigma_0 < b$. Let θ be the least upper bound of the real parts of the singularities of $g_w(s)$, and suppose that $g_w(s)$ is analytic at the real point $s = \theta$. Then, for any $\epsilon > 0$,

$$\log F_w(x) - Bx^{-b} = \Omega_\pm(x^{-(\theta-\epsilon)}) \quad \textit{as } x \to 0.$$

Proof. Let ϵ be given and consider the function

$$\Phi(x) = \log F_w(x) - Bx^{-b} - x^{-(\theta-\epsilon)}$$

with Mellin transform

$$\phi(s) = \int_0^1 x^{s-1}\Phi(x)\,dx.$$

Let σ_a be the abscissa of absolute convergence of $\phi(s)$. By Lemma 2, we have for $\operatorname{Re} s > b$

$$\phi(s) = \Gamma(s)\zeta(1+s)f_w(s) - h_w(s) - \frac{B}{s-b} - \frac{1}{s-(\theta-\epsilon)}, \tag{3.8}$$

where $h_w(s)$ is an entire function. By the definition of θ, for any $\delta > 0$, the right-hand side of (3.8) has singularities on the half-plane $\{s : \operatorname{Re} s > \theta - \delta\}$; thus we have

$$\sigma_a \geq \theta. \tag{3.9}$$

Now, if $\Phi(x) \leq 0$ for all sufficiently small x, then Lemma 3 implies that the real point $s = \sigma_a$ is a singularity of $\phi(s)$. But by the assumption of Lemma 5, $\phi(s)$ is analytic at the real point $s = \theta$ and has no singularities to the right of $s = \theta$. Thus $\sigma_a < \theta$, which contradicts (3.9). It follows that $\Phi(x)$ changes sign infinitely often, i.e., $\log F_w(x) - Bx^{-b} = \Omega_+(x^{-(\theta-\epsilon)})$. A similar argument gives $\log F_w(x) - Bx^{-b} = \Omega_-(x^{-(\theta-\epsilon)})$. □

4. Proof of Theorems 1-3

We begin by proving two lemmas which show that the function $p_\Lambda(n)$ in our theorems can be replaced by its summatory function $P_\Lambda(u) = \sum_{n\leq u} p_\Lambda(n)$.

Lemma 6. *For any integer $k \geq 15$ and $n \geq 0$, we have $p_\Lambda(n+k) \geq p_\Lambda(n)$.*

Proof. We first observe that, for any odd prime power q,

$$\sum_{n=0}^{\infty}(p_\Lambda(n+q) - p_\Lambda(n))x^n + \sum_{j=0}^{q-1} p_\Lambda(j)x^{-q+j}$$
$$= (x^{-q}-1)\prod_{m=1}^{\infty}(1-x^m)^{-\Lambda(m)} = x^{-q}(1-x^q)^{-(\Lambda(q)-1)}\prod_{m\neq q}(1-x^m)^{-\Lambda(m)}.$$

Since $\Lambda(q) - 1 > 0$ when q is an odd prime power, expanding each factor in the last expression yields a series with non-negative coefficients. Thus we have $p_\Lambda(n+q) \geq p_\Lambda(n)$ for all $n \geq 0$. Iterating this inequality with $q = 3$ and $q = 5$, we see that $p_\Lambda(n + 3k + 5l) \geq p_\Lambda(n)$ for all non-negative integers n, k and l. Since every integer ≥ 15 can be written as a linear combination of 3 and 5 with non-negative coefficients, the lemma follows. □

Lemma 7. *Let $n \in \mathbf{N}$. Let $r(u)$ be a function satisfying $r(u)(\log u)^{-1} \to \infty$ as $u \to \infty$, and $r(u+v) \ll r(u)$ uniformly for all sufficiently large u and $0 \leq v \leq 15$. Then*

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2} + O(r(n))$$

as $n \to \infty$ if and only if

$$\log P_\Lambda(u) = 2\sqrt{\zeta(2)}u^{1/2} + O(r(u))$$

as $u \to \infty$. A similar statement holds if O is replaced by $\Omega_\pm$.

Proof. By Lemma 6, we have

$$p_\Lambda(n) \leq P_\Lambda(n) = \sum_{k=0}^{n} p_\Lambda(k) \leq (n+1)p_\Lambda(n+15) \tag{4.1}$$

for all integers $n \geq 0$. Suppose that, for some positive constant C,

$$2\sqrt{\zeta(2)}u^{1/2} - Cr(u) \leq \log P_\Lambda(u) \leq 2\sqrt{\zeta(2)}u^{1/2} + Cr(u)$$

for all sufficiently large u. Then (4.1) implies that

$$2\sqrt{\zeta(2)}(n-15)^{1/2} - \log(n-14) - Cr(n-15) \leq \log p_\Lambda(n) \leq 2\sqrt{\zeta(2)}n^{1/2} + Cr(n)$$

for all sufficiently large integers n. Hence, by the assumptions on $r(u)$, we obtain

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2} + O(r(n)).$$

The proof of the converse implication and of the analogous implications between $\Omega_\pm$-estimates is similar. $\square$

By Lemma 7, it suffices to prove Theorems 1-3 with $P_\Lambda(u)$ in place of $p_\Lambda(n)$.

Proof of Theorem 1. By the Vinogradov-Korobov zero-free region for the Riemann zeta function (Titchmarsh [15, p. 135]), we have $\zeta(s) \neq 0$ for $s = \sigma + it$ satisfying

$$\sigma \geq \eta(t) = 1 - \frac{C}{(\log t^*)^{2/3}(\log_2 t^*)^{1/3}},$$

where $\log_k$ is the k times iterated logarithm, $t^* = \max(10, |t|)$, and C is a positive constant. Furthermore, in the same region we have

$$\frac{\zeta'(s)}{\zeta(s)} \ll \log t^*, \tag{4.2}$$

provided $|s-1| \geq \delta$ for some positive constant δ.

Applying Lemma 1 with $w(n) = \Lambda(n)$ and noting that in this case $f_w(s) = -\zeta'(s)/\zeta(s)$, we obtain, for $0 < x < 1$,

$$\log F_\Lambda(x) = -\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)}x^{-s}\,ds + O\left(e^{-T}\frac{\zeta'(\kappa)}{\zeta(\kappa)}x^{-\kappa}\right),$$

where $1 < \kappa < 2$ and $T > 10$ are positive numbers to be chosen later. We shift the path of integration to the path consisting of the segments $\gamma_1 : \{\sigma - iT : \kappa \geq \sigma > \eta(T)\}$, $\gamma_2 : \{\eta(t) + it : -T \leq t < -10\}$, $\gamma_3 : \{\eta(t) + it : -10 \leq t < 10\}$, $\gamma_4 : \{\eta(t) + it : 10 \leq t < T\}$ and $\gamma_5 : \{\sigma + iT : \eta(t) \leq \sigma \leq \kappa\}$.

The estimate

$$|\Gamma(s)| \ll (1+|t|)^{\sigma+1/2}e^{-\pi|t|/2} \ll e^{-|t|} \qquad (\frac{1}{2} \leq \sigma \leq 2,\ t \in \mathbf{R})$$

(which follows from Stirling's formula) and (4.2) imply that

$$\int_{\gamma_{1,5}} \ll e^{-T}(\log T)x^{-\kappa}$$

and

$$\int_{\gamma_{2,4}} \ll \int_{10}^{T} e^{-t}(\log t)x^{-\eta(t)}\,dt \ll x^{-\eta(T)}.$$

We also have

$$\int_{\gamma_3} \ll x^{-\eta(10)}.$$

Thus, we obtain

(4.3)

$$\log F_\Lambda(x) = \zeta(2)x^{-1} + O\left(e^{-T}\frac{\zeta'(\kappa)}{\zeta(\kappa)}x^{-\kappa}\right) + O\left(e^{-T}(\log T)x^{-\kappa}\right) + O\left(x^{-\eta(T)}\right).$$

We now assume that x is sufficiently small so that $\log_3 x^{-1}$ is defined and ≥ 1. We set $\kappa = 1 + 1/\log x^{-1}$ and

$$T = \frac{C\log x^{-1}}{(\log_2 x^{-1})^{2/3}(\log_3 x^{-1})^{1/3}},$$

where C is the constant appearing in the definition of $\eta(t)$. Then $x^{-\kappa} = ex^{-1}$ and $\zeta'(\kappa)/\zeta(\kappa) \ll \log x^{-1}$. Hence, noting that

$$1 - \eta(T) \sim \frac{C}{(\log_2 x^{-1})^{2/3}(\log_3 x^{-1})^{1/3}}$$

as $x \to 0$, we obtain from (4.3)

$$\log F_\Lambda(x) = \zeta(2)x^{-1} + O\left(x^{-1}\exp\left\{-\frac{(C/2)\log x^{-1}}{(\log_2 x^{-1})^{2/3}(\log_3 x^{-1})^{1/3}}\right\}\right). \tag{4.4}$$

We now apply Proposition 2 with $B = B' = \zeta(2)$, $b = 1$ and the quantities A and a determined by (2.3), i.e.,

$$a = 1/2 \qquad \text{and} \qquad A = a^{-1}\left(Ba/(1-a)\right)^{1-a} = 2\sqrt{\zeta(2)}. \tag{4.5}$$

Setting $y = B'x^{-b-1} = \zeta(2)x^{-2}$ and

$$r(y) = y^{1/2}\exp\left\{-\frac{(C/5)\log y}{(\log_2 y)^{2/3}(\log_3 y)^{1/3}}\right\},$$

it easily follows from (4.4) that

$$|\log F_\Lambda(x) - \zeta(2)x^{-1}| \leq r(\zeta(2)x^{-2})$$

for all sufficiently small x. Hence, by Proposition 2, we may conclude that, with $c = C/10$,

$$\begin{aligned}\log P_\Lambda(u) &= Au^a + O(\sqrt{r(u)u^a}) \\ &= 2\sqrt{\zeta(2)}u^{1/2} + O\left(u^{1/2}\exp\left\{-\frac{c\log u}{(\log_2 u)^{2/3}(\log_3 u)^{1/3}}\right\}\right).\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. We will prove only the Ω_--estimate; the Ω_+-estimate can be proved similarly.

Applying the corollary to Proposition 1 with $P(u) = P_\Lambda(u)$, $F(x) = F_\Lambda(x)$, $A = 2\sqrt{\zeta(2)}$, $a = 1/2$, $B = B' = \zeta(2)$, $b = 1$ and $r(u) = u^{1/4}$, we see that it suffices to show that

$$\log F_\Lambda(x) - \zeta(2)x^{-1} = \Omega_-(x^{-1/2}). \tag{4.6}$$

Let θ be the least upper bound of the real parts of the zeros of the Riemann zeta function. If the Riemann Hypothesis is false, i.e., if $\theta > 1/2$, we apply Lemma 5 with $w(n) = \Lambda(n)$, $f_w(s) = -\zeta'(s)/\zeta(s)$, $B = \zeta(2)$ and $b = 1$. The function $g_w(s)$ in Lemma 5 then is given by $-\zeta'(s)/\zeta(s) - 1/(s-1)$. Thus $g_w(s)$ is a meromorphic function in $\operatorname{Re} s > 0$, whose singularities are exactly the zeros of $\zeta(s)$. In particular, the least upper bound of the real parts of these singularities is equal to the least upper bound θ of the real parts of the zeros of $\zeta(s)$. Moreover, since $\zeta(s)$ has no zeros on the real line, $g_w(s)$ is analytic at $s = \theta$. Thus, Lemma 5 implies that, for any $\epsilon > 0$,

$$\log F_\Lambda(x) - \zeta(2)x^{-1} = \Omega_\pm(x^{-(\theta-\epsilon)}).$$

In particular, we can choose $\epsilon = \theta - 1/2$ and the assertion of Theorem 2 follows. We therefore assume that the Riemann Hypothesis is true.

Let $1/2 + i\gamma$ be one of the non-trivial zeros of the Riemann zeta function and let m be its multiplicity. Let C be a positive constant such that

$$C < m|\Gamma(1/2 + i\gamma)\zeta(3/2 + i\gamma)| \tag{4.7}$$

and define

$$\Phi(x) = \log F_\Lambda(x) - \zeta(2)x^{-1} + Cx^{-1/2}.$$

If (4.6) is false, then there exists $x_0 > 0$ such that $\Phi(x) \geq 0$ holds for all $x \leq x_0$. By Lemma 3, the integral

$$\phi(s) = \int_0^1 \Phi(x)x^{s-1}\,dx$$

has a singularity at the real point $s = \sigma_a$, the abscissa of absolute convergence of $\phi(s)$. On the other hand, applying Lemma 2 with $f_w(s) = -\zeta'(s)/\zeta(s)$, we obtain

$$\phi(s) = -\Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)} - h_\Lambda(s) - \frac{\zeta(2)}{s-1} + \frac{C}{s-1/2}, \tag{4.8}$$

where $h_\Lambda(s)$ is an entire function. Since the last expression has no singularity on the real axis to the right of $1/2$, the abscissa of absolute convergence σ_a of the integral $\phi(s)$ must be $\leq 1/2$. For $\sigma > 1/2$ we have

$$\begin{aligned} |\phi(\sigma + i\gamma)| &\leq \int_{x_0}^1 |\Phi(x)|x^{\sigma-1}\,dx + \int_0^{x_0} \Phi(x)x^{\sigma-1}\,dx \\ &= \int_{x_0}^1 \{|\Phi(x)| - \Phi(x)\}x^{\sigma-1}\,dx + \int_0^1 \Phi(x)x^{\sigma-1}\,dx \\ &\leq \int_{x_0}^1 2|\Phi(x)|x^{-1/2}\,dx + \phi(\sigma). \end{aligned} \tag{4.9}$$

We multiply both sides by $\sigma - 1/2$ and let $\sigma \to 1/2+$. The right-hand side becomes $\lim_{\sigma\to 1/2+}(\sigma - 1/2)\phi(\sigma)$, which, by (4.8), is equal to

$$\lim_{\sigma\to 1/2+}(\sigma - 1/2)\left| -\Gamma(\sigma)\zeta(1+\sigma)\frac{\zeta'(\sigma)}{\zeta(\sigma)} - \frac{\zeta(2)}{\sigma-1} + \frac{C}{\sigma-1/2}\right| = C.$$

On the other hand, applying (4.8) to $\phi(\sigma + i\gamma)$, we see that the left-hand side of (4.9) becomes (after multiplying by $\sigma - 1/2$ and letting $\sigma \to 1/2+$)

$$\begin{aligned} \lim_{\sigma\to 1/2+}(\sigma - 1/2)\Bigg| &-\Gamma(\sigma + i\gamma)\zeta(1+\sigma+i\gamma)\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)} \\ &- \frac{\zeta(2)}{\sigma-1+i\gamma} + \frac{C}{\sigma-1/2+i\gamma}\Bigg| = m|\Gamma(1/2+i\gamma)\zeta(3/2+i\gamma)|. \end{aligned}$$

Hence $m|\Gamma(1/2 + i\gamma)\zeta(3/2 + i\gamma)| \leq C$, which contradicts (4.7). Therefore (4.6) holds, and the proof of Theorem 2 is complete. □

Proof of Theorem 3. Let θ be the least upper bound for the real parts of the zeros of the Riemann zeta function. It suffices to show that if α is a real number such that the estimate

$$\log P_\Lambda(u) = 2\sqrt{\zeta(2)}u^{1/2} + O(u^{\alpha/2}) \tag{4.10}$$

holds as $u \to \infty$, then $\alpha \geq \theta$.

By Proposition 1, (4.10) implies that

$$\log F_\Lambda(x) = \zeta(2)x^{-1} + O(x^{-\alpha}).$$

Thus the integral

$$\phi(s) = \int_0^1 x^{s-1}\left\{\log F_\Lambda(x) - \zeta(2)x^{-1}\right\}\, dx$$

defines an analytic function on the half-plane $\{s : \operatorname{Re} s > \alpha\}$. On the other hand, by Lemma 2 we have, for $\operatorname{Re} s > 1$,

$$\phi(s) = -\Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta(2)}{s-1} - h_\Lambda(s), \tag{4.11}$$

where $h_\Lambda(s)$ is an entire function. Since $\phi(s)$ is analytic in $\operatorname{Re} s > \alpha$, the latter relation remains valid in this larger region, and the right-hand side of (4.11) is analytic in $\operatorname{Re} s > \alpha$. Therefore $\zeta(s)$ cannot have zeros on the half-plane $\{s : \operatorname{Re} s > \alpha\}$, i.e., we have $\theta \leq \alpha$, as claimed. □

5. A New Proof of Theorem B

Let θ be the least upper bound for the real parts of the zeros of the Riemann zeta function. We may assume $1/2 \leq \theta < 1$, for if $\theta = 1$, the asserted estimate follows from Theorem 1.

By Lemma 1 we have, for $0 < x < 1$ and $\kappa > 1$,

$$\log F_\Lambda(x) = -\frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} \Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)}x^{-s}\, ds.$$

By standard arguments (as, for example, in the proof of Theorems 28 of Ingham [5]) we now shift the line of integration to the vertical line $\operatorname{Re} s = -1/2$, taking into account the residues of the integrand at $s = 1$, 0 and at the zeta-zeros. This gives

$$\begin{aligned}\log F_\Lambda(x) = \zeta(2)x^{-1} &- \sum_\rho \Gamma(\rho)\zeta(1+\rho)x^{-\rho} - \frac{\zeta'(0)}{\zeta(0)}\log x^{-1} \\ &- \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)}x^{-s}\, ds,\end{aligned}$$

where ρ runs over non-trivial zeta-zeros, counted with multiplicities.

By the symmetry of the zeta-zeros and the assumption that $\theta < 1$, we have

$$|\zeta(1+\rho)| \leq \zeta(1 + (1-\theta)) \ll 1$$

for all zeta-zeros ρ. Using Stirling's formula and the bound

$$\sum_{T \leq \operatorname{Im}\rho \leq T+1} 1 \ll \log T,$$

we obtain

$$\sum_{\rho} \Gamma(\rho)\zeta(1+\rho)x^{-\rho} \ll \sum_{\rho=\beta+i\gamma} e^{-|\gamma|}x^{-\beta} \ll x^{-\theta}.$$

Also, for $\operatorname{Re} s = -1/2$ we have $\zeta'(s)/\zeta(s) \ll \log(|s|+2)$ and $\zeta(1+s) \ll (|s|+1)^{1/2}$ (Theorems 9 and 27 in [5]). Thus

$$\begin{aligned}\int_{-1/2-i\infty}^{-1/2+i\infty} &\Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)}x^{-s}\,ds \\ &\ll x^{1/2}\int_{-\infty}^{\infty} e^{-|t|}(|t|+1)^{1/2}\log(|t|+2)\,dt \ll x^{1/2}.\end{aligned}$$

Hence, we obtain

$$\log F_\Lambda(x) = \zeta(2)x^{-1} + O(x^{-\theta}). \tag{5.1}$$

Similarly, we have, for $\kappa > 1$,

$$\begin{aligned}\frac{d}{dx}\log F_\Lambda(x) &= \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} s\Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)}x^{-s-1}\,ds \\ &= -\zeta(2)x^{-2} + O(x^{-(\theta+1)})\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\frac{d^2}{dx^2}\log F_\Lambda(x) &= -\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} s(1+s)\Gamma(s)\zeta(1+s)\frac{\zeta'(s)}{\zeta(s)}x^{-s-2}\,ds \\ &= 2\zeta(2)x^{-3} + O(x^{-(\theta+2)}).\end{aligned} \tag{5.3}$$

By (5.1)-(5.3), the function $F(x) = F_\Lambda(x)$ satisfies the hypotheses (2.15), (2.22) and (2.23) of Proposition 3 with $b = 1$, $B = B' = \zeta(2)$ and $r(u) = Ku^{\theta/2}$, where K is a sufficiently large constant. Moreover, the function $r(u)$ satisfies conditions (R1), (R2) and (R3) of that proposition. Hence the conclusion (2.24) of Proposition 3 holds with A and a given by (4.5). Therefore

$$\log P_\Lambda(u) = Au^a + O(r(u)) = 2\sqrt{\zeta(2)}u^{1/2} + O(u^{\theta/2})$$

as $u \to \infty$, which is the claimed result.

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